Disuguaglianza di Poincaré

Teorema 1. *Siano $B_R \subset \mathbb{R}^d$ e $u \in H^1(B_R)$. Allora,*

$$\frac{1}{R^2}\int_{B_R} \big|u(x) - M_R\big|^2\,dx \le C_d \int_{B_R} |\nabla u|^2\,dx \qquad \text{dove} \qquad M_R := \fint_{B_R} u(x)\,dx\ ,$$

e dove C_d è una costante dimensionale.

Proof. Sia $y \in B_R$. Definiamo

$$\varphi(x) := u(x-y)\ , \qquad \varphi : B_R(-y) \to \mathbb{R},$$

e osserviamo che possiamo scrivere l'insieme $B_R(-y)$ in coordiante polari come

$$B_R(-x_0) := \Big\{(r,\theta)\ :\ \theta \in \partial B_1, \quad r \in [0, R_\theta)\Big\}.$$

Per ogni

$$x = (r,\theta) \in B_R(-y),$$

abbiamo la stima

$$|\varphi(r\theta) - \varphi(0)| \le \left|\int_0^r \theta\cdot\nabla\varphi(s\theta)\,ds\right| \le \int_0^r |\nabla\varphi|(s\theta)\,ds.$$

Integrando in r e in θ abbiamo

$$\begin{aligned}
\int_{B_R} |u(x) - u(y)|^2\,dx &= \int_{B_R(-y)} |\varphi - \varphi(0)|^2 \\
&= \int_{\partial B_1}\int_0^{R_\theta} r^{d-1}|\varphi(r\theta) - \varphi(0)|^2\,dr\,d\theta \\
&\le \int_{\partial B_1}\int_0^{R_\theta} r^{d-1}\left(\int_0^r |\nabla\varphi|(s\theta)\,ds\right)^2\,dr\,d\theta \\
&\le \int_{\partial B_1}\int_0^{R_\theta} r^{d}\int_0^r |\nabla\varphi|^2(s\theta)\,ds\,dr\,d\theta \\
&= \int_{\partial B_1}\int_0^{R_\theta} r^{d}\int_0^{R_\theta} |\nabla\varphi|^2(s\theta)\,ds\,dr\,d\theta \\
&\le \frac{(2R)^{d+1}}{d+1}\int_{\partial B_1}\int_0^{R_\theta} |\nabla\varphi|^2(s\theta)\,ds\,d\theta,
\end{aligned}$$

dove nell'ultima disuguaglianza abbiamo usato che

$$\int_0^{R_\theta} r^d\,dr \le \int_0^{2R} r^d\,dr = \frac{1}{d+1}(2R)^{d+1}.$$

Ora, osserviamo che

$$\int_{\partial B_1}\int_0^{R_\theta} |\nabla\varphi|^2(s\theta)\,ds\,d\theta = \int_{\partial B_1}\int_0^{R_\theta} \frac{|\nabla\varphi|^2(s\theta)}{s^{d-1}}\,s^{d-1}ds\,d\theta = \int_{B_R(-y)} \frac{|\nabla\varphi|^2(x)}{|x|^{d-1}}\,dx = \int_{B_R} \frac{|\nabla u|^2(x)}{|x-y|^{d-1}}\,dx$$

Quindi abbiamo

$$\int_{B_R} |u(x) - u(y)|^2\,dx \le \frac{2^{d+1}}{d+1}R^{d+1}\int_{B_R} \frac{|\nabla u|^2(x)}{|x-y|^{d-1}}\,dx$$

Integrando in y, abbiamo

$$\begin{aligned}
\int_{B_R}\int_{B_R} |u(x) - u(y)|^2\,dx\,dy &\le \frac{2^{d+1}}{d+1}R^{d+1}\int_{B_R}\int_{B_R} \frac{|\nabla u|^2(x)}{|x-y|^{d-1}}\,dx\,dy \\
&\le \frac{2^{d+1}}{d+1}R^{d+1}\int_{B_R} \frac{1}{|x-y|^{d-1}}\,dy\int_{B_R} |\nabla u|^2\,dx.
\end{aligned}$$

Siccome

$$\int_{B_R} \frac{1}{|x-y|^{d-1}}\,dy \le \int_{B_R} \frac{1}{|y|^{d-1}}\,dy = d\omega_d R,$$

e $|B_R| = \omega_d R^d$ otteniamo

$$\int_{B_R}\int_{B_R} |u(x)-u(y)|^2\,dx\,dy \le \frac{d2^{d+1}}{d+1}|B_R|R^2\int_{B_R}|\nabla u|^2\,dx.$$

Infine, usando l'identità

$$\frac{1}{|B_R|}\int_{B_R}\int_{B_R}|u(x)-u(y)|^2\,dx\,dy = 2\int_{B_R}\big|u(x)-M_R\big|^2.$$

dove

$$M_R := \fint_{B_R} u(x)\,dx$$

otteniamo

$$\int_{B_R}\big|u(x)-M_R\big|^2\,dx \le C_d R^2\int_{B_R}|\nabla u|^2\,dx,$$

con $C_d = \frac{d}{d+1}2^d$. $\square$